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Abstract

Using the work by Bampi and Caviglia, we write the Weyl-Lanczos equations as an exterior differential system. Using Janet-Riquier theory, we compute the Cartan characters for all spacetimes with a diagonal metric and for the plane wave spacetime since all spacetimes have a plane wave limit.

We write the Lanczos wave equation as an exterior differential system and, with assistance from Janet-Riquier theory, we find that it forms a system in involution. This result can be derived from the scalar wave equation itself. We compute its Cartan characters and compare them with those of the Weyl-Lanczos equations.

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I. Introduction

A. The Weyl-Lanczos equations and the Lanczos Tensor Wave Equation in 4 Dimensions

Lanczos [?] generated the spacetime Weyl conformal tensor C_{abcd} from a tensor potential L_{abc} by covariant differentiation such that C_{abcd} is given by

$$\begin{aligned} C_{abcd} &= L_{abc;d} - L_{abd;c} + L_{cda;b} - L_{cdb;a} + g_{bc}L_{(ad)} + g_{ad}L_{(bc)} - g_{bd}L_{(ac)} \\ &\quad - g_{ac}L_{(bd)} + \frac{2}{3}L^{ms}{}_{m;s}(g_{ac}g_{bd} - g_{ad}g_{bc}) \,, \end{aligned} \tag{1}$$

where $L_{(ad)} = L_a{}^s{}_{d;s} - L_a{}^s{}_{s;d}$ and $'';''$ denotes covariant differentiation. We call (1) the Weyl-Lanczos equations. The index symmetries of the Lanczos tensor L_{abc} have to match the symmetries of (1) and so it is usual to impose

$$L_{abc} = L_{[ab]c} \,, \tag{2}$$

and

$$L_{[abc]} = 0 \tag{3}$$

and the trace free (gauge) condition

$$L_a{}^s{}_s = 0 \,. \tag{4}$$

The spacetime Weyl Lanczos equations (1) can also be expressed in the more compact form

$$C_{abcd} = L_{[ab][c;d]} + L_{[cd][a;b]} -^* L^*_{[ab][c;d]} -^* L^*_{[cd][a;b]} \tag{5}$$

as done in [?], where “;$''$ denotes covariant differentiation. The algebraic equations (2), (3), (4) leave us with only 16 independent components for the L_{abc}. If we then introduce the differential gauge conditions

$$L_{ab}{}^s{}_{;s} = 0 \,, \tag{6}$$

we can simplify (1) considerably to get

$$C_{abcd} = L_{abc;d} - L_{abd;c} + L_{cda;b} - L_{cdb;a} - g_{bc}L^s_{ad;s} - g_{ad}L^s_{bc;s} + g_{bd}L^s_{ac;s} + g_{ac}L^s_{bd;s} \,. \tag{7}$$

Theoretically, we could completely solve the 6 differential gauge conditions for 6 further components and have 10 components for 10 independent spacetime Weyl-Lanczos equations. But this approach does not exhibit the most general solution. We note that equations (6), (7) constitute a system of linear first-order partial differential equations in 4 dimensions which can easily be rewritten as an exterior differential system EDS in involution. This theory only applies in 4 dimensions as the Weyl Lanczos problem in 2 and 3 dimensions does not exist and for 5 dimensions, we expect extra conditions to apply as we shall point out in a later paper. It was already shown in [?] that the above 4-dimensional Weyl-Lanczos equations consist of an exterior differential system (EDS) in involution with respect to the spacetime variables. One can obtain the same results by applying the corresponding Janet-Riquier theory [?].

From the Weyl-Lanczos problem it is possible to generate a tensor wave equation for the (spacetime) Lanczos potential from which the Penrose wave equation for the Weyl tensor C_{abcd} can be derived [?]. Arising from the Weyl-Lanczos equations is the linear tensor wave equation

$$\Box L_{abc}+2R_c^{\ s}L_{abs}-R_a^{\ s}L_{bcs}-R_b^{\ s}L_{cas}-g_{ac}R^{ls}L_{lbs}+g_{bc}R^{ls}L_{las}-\frac{1}{2}RL_{abc}=J_{abc}, \tag{8}$$

where

$$J_{abc}=\frac{1}{2}R_{c[a;b]}-\frac{1}{6}g_{c[a}R_{;b]} \tag{9}$$

and

$$\Box L_{abc}=g^{sm}L_{abc;s;m}\ , \tag{10}$$

from which Penrose's non-linear wave equation[1] for the spacetime Weyl tensor

$$\begin{aligned}\Box C_{abcd}-C_{ab}{}^{sm}C_{smcd}+4C_{asm[c}C^{m}{}_{d]}{}^{s}{}_{b}+\frac{R}{4}C_{abcd} &= J_{[ab][c;d]}+J_{[cd][a;b]} \\ &\quad -^{*}J^{*}_{[ab][c;d]}-^{*}J^{*}_{[cd][a;b]}\end{aligned} \tag{11}$$

was derived in [?].

It was shown by Bampi and Caviglia [?] that the 5 dimensional version of equation (1) with the appropriate equations generalising (2), (3), (4), (6) constitutes a system of linear first order partial differential equations which is also an EDS in involution. In this paper we are going to show that like the Weyl-Lanczos problem (1) the spacetime Lanczos tensor wave equation (8) can be rewritten as an EDS in involution, a result which is confirmed using Janet-Riquier theory.

II. The Weyl-Lanczos Equations in 4 Dimensions as a System in Involution

In [?] it was shown that the Weyl-Lanczos relations are a system in involution with respect to the spacetime variables. The Weyl-Lanczos equations always constitute of a system in involution as opposed to the Riemann-Lanczos equations, even for vacuum spacetimes when $C_{abcd}=R_{abcd}$, because each problem is based on different equations.

A. The Weyl-Lanczos Equations as an EDS

The theory of exterior differential systems (EDS) can be found in many places such as in [?, ?, ?, ?]. A review together with some results on the Riemann-Lanczos problem in 4 dimensions is given in [?].

The EDS for the Weyl-Lanczos equations, which form a Pfaffian system, was already given in [?]. The Weyl-Lanczos equations together with the differential gauge condition (6) form a system of 16 linear first-order PDEs. In a local coordinate frame, these 16 equations look like

$$\begin{aligned}
f_{abcd} &= C_{abcd}-P_{abcd}+P_{abdc}-P_{cdab}+P_{cdba}+\Gamma^n_{ad}(L_{nbc}+L_{ncb}) \\
&\quad -\Gamma^n_{ac}(L_{nbd}+L_{ndb})+\Gamma^n_{bc}(L_{nad}+L_{nda})-\Gamma^n_{bd}(L_{nac}+L_{nca}) \\
&\quad +g_{bc}g^{ns}P_{nads}+g_{ad}g^{ns}P_{nbcs}-g_{bd}g^{ns}P_{nacs}-g_{ac}g^{ns}P_{nbds} \\
&\quad -g_{bc}g^{ns}(\Gamma^m_{ns}L_{mad}+\Gamma^m_{as}L_{nmd}+\Gamma^m_{ds}L_{nam})-g_{ad}g^{ns}(\Gamma^m_{ns}L_{mbc} \\
&\quad +\Gamma^m_{bs}L_{nmc}+\Gamma^m_{cs}L_{nbm})+g_{bd}g^{ns}(\Gamma^m_{ns}L_{mac}+\Gamma^m_{as}L_{nmc}+\Gamma^m_{cs}L_{nam}) \\
&\quad +g_{ac}g^{ns}(\Gamma^m_{ns}L_{mbd}+\Gamma^m_{bs}L_{nmd}+\Gamma^m_{ds}L_{nbm})\ , \\
f_{ab} &= g^{ns}(P_{abns}-\Gamma^m_{as}L_{mbn}-\Gamma^m_{bs}L_{amn}-\Gamma^m_{ns}L_{abm})\ ,
\end{aligned} \tag{12}$$

where $f_{abcd} = 0$ denotes the Weyl-Lanczos equations and $f_{ab} = L_{ab}{}^{s}{}_{;s} = 0$ the differential gauge conditions in local coordinates. When we construct the corresponding EDS, we introduce the local coordinates (x^e, L_{abc}, P_{abcd}) on the jet bundle $\mathcal{J}^1(\mathbb{R}^4, \mathbb{R}^{16})$ which form our formal manifold $\mathcal{M}$ of formal dimension $N = 4 + 16 + 64 = 84$.

The exterior derivatives of all equations in (12) constitute our first 16 one-forms. We also have to add 16 **contact conditions** K_{abc} in order to make sure that the P_{abcd} can be considered as if they were the partial derivatives of the L_{abc}. These additional 16 one-forms are locally given by $K_{abc} = dL_{abc} - P_{abce}dx^e$. In this way we obtain the Pfaffian system $\mathcal{P}$

$$
\begin{aligned}
df_{abcd} &= [C_{abcd,e} + \alpha_{abcde} + \gamma_{abcde}]dx^e - dP_{abcd} + dP_{abdc} - dP_{cdab} + dP_{cdba} \\
&\quad + g_{bc}g^{ns}dP_{nads} + g_{ad}g^{ns}dP_{nbcs} - g_{bd}g^{ns}dP_{nacs} - g_{ac}g^{ns}dP_{nbds} \\
&\quad + \Gamma^n_{ad}(dL_{nbc} + dL_{ncb}) - \Gamma^n_{ac}(dL_{nbd} + dL_{ndb}) + \Gamma^n_{bc}(dL_{nad} + dL_{nda}) \\
&\quad - \Gamma^n_{bd}(dL_{nac} + dL_{nca}) - g_{bc}g^{ns}(\Gamma^m_{ns}dL_{mad} + \Gamma^m_{as}dL_{nmd} + \Gamma^m_{ds}dL_{nam}) \\
&\quad - g_{ad}g^{ns}(\Gamma^m_{ns}dL_{mbc} + \Gamma^m_{bs}dL_{nmc} + \Gamma^m_{cs}dL_{nbm}) + g_{bd}g^{ns}(\Gamma^m_{ns}dL_{mac} \\
&\quad + \Gamma^m_{as}dL_{nmc} + \Gamma^m_{cs}dL_{nam}) + g_{ac}g^{ns}(\Gamma^m_{ns}dL_{mbd} + \Gamma^m_{bs}dL_{nmd} \\
&\quad + \Gamma^m_{ds}dL_{nbm}), \\
df_{ab} &= (P_{abns}g^{ns}{}_{,e} - L_{mbn}\Gamma^m_{as,e} - L_{amn}\Gamma^m_{bs,e} - L_{abm}\Gamma^m_{ns,e})dx^e \\
&\quad + g^{ns}(dP_{abns} - \Gamma^m_{as}dL_{mbn} - \Gamma^m_{bs}dL_{amn} - \Gamma^m_{ns}dL_{abm}), \\
K_{abc} &= dL_{abc} - P_{abce}dx^e\,,
\end{aligned}
\tag{13}
$$

where α_{abcde} and γ_{abcde} are given in Appendix A. Now, a Vessiot vector field for (13), where 16 of the totally 64 V_{abcd} are determined through $df_{abcd}(V) = 0, df_{ab}(V) = 0$ from (13), will be of the form

$$
V = V^e \frac{\partial}{\partial x^e} + V^e P_{abce}\frac{\partial}{\partial L_{abc}} + V_{abcd}\frac{\partial}{\partial P_{abcd}}\,. \tag{14}
$$

Because $K_{abc}(V) = 0$ has to hold as well, we get $V_{abc} = V^e P_{abce}$ as used above. We can determine the associated system $\mathcal{A}(\mathcal{P})$ of (13) which is given locally by

$$
\{df_{abcd}, dL_{ab}{}^{s}{}_{;s}, \omega^e, dP_{abcd}\}\,,
$$

where $\omega^e := dx^e$ and where we only include 48 of the 64 dP_{abcd}. This is because 16 of them can be expressed by means of the exterior derivatives of the Weyl-Lanczos relations and the differential gauge equations. We see that $\dim(\mathcal{A}(\mathcal{P})) = 84 = c$, the class of $\mathcal{P}$, so that $\dim(\bar{\mathcal{D}})$=0, where $\bar{\mathcal{D}}$ is defined as

$$
\bar{\mathcal{D}} = \{Y \in \mathcal{D} / d\theta^\alpha(X, Y,) = 0\, \forall X \in \mathcal{D}\, \forall 1 \leq \alpha \leq s\}\,, \tag{15}
$$

where s denotes to the total number of independent 1-forms θ^α in our Pfaffian system $\mathcal{P}$. If we choose $Y \in Char(\mathcal{P})$, we have to be able to express

$$
Y \rfloor dK_{abc} = Y^e dP_{abce} - Y_{abce}dx^e
$$

by means of a non-trivial linear combinations of the forms $K_{abc}, df_{ab}, df_{abcd}$ with linear multipliers $\lambda^{a'b'c'}$,
$\lambda^{a'b'c'd'}, \lambda^{nm}$

$$
\lambda^{a'b'c'}K_{a'b'c'} + \lambda^{a'b'c'd'}df_{a'b'c'd'} + \lambda^{mn}dL_{mn}{}^{s}{}_{;s}\,. \tag{16}
$$

However, no such linear combination can exist because some of the dP_{abcd} will fail to occur in one or other of the df_{abcd} or the df_{ab} and in this way we know that *no Cauchy characteristics can occur.*

Next, we want to compute the reduced Cartan characters by hand for this Pfaffian system $\mathcal{P}$ using the tableau matrix. First, we must complete $(df_{abcd}, df_{ab}, \omega^e)$ so that it becomes a complete coframe on $\mathcal{M}$, say $(df_{abcd}, df_{ab}, K_{abc}, \omega^e, \pi^\Lambda)$, where we use the forms in (13) as cobasis elements. Accordingly, we must add the 48 new cobasis elements

$$
\begin{array}{lll}
\pi^\Lambda \leftrightarrow & dP_{abc1}, & \Lambda = 1, \cdots, 16\,, \\
\pi^\Lambda \leftrightarrow & dP_{abc2}, & \Lambda = 17, \cdots, 32\,, \\
\pi^\Lambda \leftrightarrow & dP_{abc3}, & \Lambda = 33, \cdots, 48\,,
\end{array}
$$

where the ordering of the P_{abcd} based on the indices abc is given in Appendix A and where Λ is a collective index subject to Einstein's summation convention and corresponding to the set of indices abc . Here, we write $\omega^1 = dx^1$, $\omega^2 = dx^2$, $\omega^3 = dx^3$, $\omega^4 = dx^4$ to form the independence condition given by $\Omega = \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4$. Using this notation we can express the 16 exterior derivatives of the contact conditions, where the α are arranged in the same way as the L_{abc} in Appendix A, as

$$
d\theta^\alpha = A^\alpha_{\Lambda 1}\pi^\Lambda \wedge \omega^1 + A^\alpha_{\Lambda 2}\pi^\Lambda \wedge \omega^2 + A^\alpha_{\Lambda 3}\pi^\Lambda \wedge \omega^3 - dP_{abc4} \wedge \omega^4\,. \tag{17}
$$

Further calculations are based on the assumption that we can express each of the 16 dP_{abc4} in (17) as a distinct linear combination of the $df_{abcd}, K_{abc}, dL^s_{ab\;;s}$ and the ω^e. Later on, it will be easy to verify that this is true for all spacetimes with diagonal metric tensor. The tableau matrices $A^\alpha_{\Lambda 2}$,$A^\alpha_{\Lambda 3}$ and $A^\alpha_{\Lambda 4}$ can now easily be determined from (17), where we obtain the only nonvanishing components to be $A^\alpha_{\Lambda 2} = A^\alpha_{\Lambda 3} = A^\alpha_{\Lambda 4} = -1$ if α and Λ refer to the same group of indices abc. We do not need to determine $A^\alpha_{\Lambda 4}$ explicitly as it does not contribute to the characters. This is because the sum of the ranks of the tableau matrices of increasing order cannot exceed 48 and so the terms in $A^\alpha_{\Lambda 4}$ do not contribute to the rank. In this way, the matrix

$$
\begin{pmatrix}
A^\alpha_{\Lambda\,1} \\
A^\alpha_{\Lambda\,2} \\
A^\alpha_{\Lambda\,3} \\
A^\alpha_{\Lambda\,4}
\end{pmatrix}
$$

is a 64x48-matrix which looks like:

$$
\begin{pmatrix}
A & 0 & 0 \\
0 & A & 0 \\
0 & 0 & A \\
X_1 & X_2 & X_3
\end{pmatrix},
$$

where X_1, X_2, X_3 stand for $A^\alpha_{\Lambda 4}$ and each A is given by the 16x16-matrix $A := -\mathbb{I}_{16}$. From this we can immediately deduce that the reduced characters are $s_1' = s_2' = s_3' = 16$ and $s_4' = 0$. However, this method does not supply s_0' but here clearly $s_0' = s = 32$ which is simply the number of independent 1-forms in (13) consisting of the 10 exterior derivatives of the Weyl-Lanczos equations, the 6 exterior derivatives of the differential gauge conditions and the 16 contact conditions K_{abc}. The reduced characters $(s_0', s_1', s_2', s_3', s_4')$ then are given by $(32, 16, 16, 16, 0)$. When (13) is pulled back onto the submanifold, where $f_{abcd} = 0$ and $f_{ab} = 0$, we get $(16, 16, 16, 16, 0)$.

Lastly, we make an Ansatz for a transformation which absorbs the remaining torsion terms B^{α}_{ij}. Torsion terms can arise when we write each $d\theta^{\alpha}$ of a Pfaffian system $\mathcal{P}$ as

$$d\theta^{\alpha} = A^{\alpha}_{\lambda i}\pi^{\lambda} \wedge \omega^{i} + \frac{1}{2}B^{\alpha}_{ij}\omega^{i} \wedge \omega^{j} + \frac{1}{2}C^{\alpha}_{\lambda\kappa}\pi^{\lambda} \wedge \pi^{\kappa} . \tag{18}$$

The only non-vanishing torsion terms here are those of type B^{α}_{i4}. In order to make them vanish, we must look for a transformation Φ of the form

$$\begin{aligned} \pi^{\Lambda} \rightarrow &\quad \pi^{\Lambda} + p^{\Lambda}_{i}\omega^{i} , \\ B^{\alpha}_{ij} \rightarrow &\quad \tilde{B}^{\alpha}_{ij} = B^{\alpha}_{ij} + A^{\alpha}_{\Lambda i}p^{\Lambda}_{j} - A^{\alpha}_{\Lambda j}p^{\Lambda}_{i} , \end{aligned}$$

where we choose the p^{Λ}_{i} in such a way that $\tilde{B}^{\alpha}_{ij} = 0$. Such p^{Λ}_{i} must be solutions to

$$0 = B^{\alpha}_{i4} - A^{\alpha}_{\Lambda 4}p^{\Lambda}_{i} , \tag{19}$$

where we can choose $p^{\Lambda}_{4} = 0$ because the B^{α}_{44} always vanish due to skew symmetry. For any such solution of the p^{Λ}_{i} the torsion is absorbed and the system is in involution. This rather cumbersome calculation can be carried out using a REDUCE code in [?] based on the package EDS [?]. We obtained the result that the Cartan characters coincide with their reduced counterparts and that the torsion can be absorbed for a number of spacetimes such as Schwarzschild, Kasner, the Debever-Hubaut class, Gödel, the pp-wave spacetimes and conformally flat spacetimes.

B. The Weyl-Lanczos Equations as a System of PDEs

The theory in this section and for the Lanczos wave equation below is based on a modernised version of Janet-Riquier theory [?, ?, ?]. In this modernised form Janet-Riquier theory can be found in [?, ?] and, a review together with some results on the Riemann-Lanczos problems in 2 and 3 dimensions is given in [?, ?].

The 10 independent Weyl-Lanczos relations and the 6 differential gauge conditions in local coordinates are given by (12). Using the first computer code in Appendix C in [?], we can derive the symbol $\mathcal{M}_1$ for any spacetime with diagonal metric

$$ds^2 = a_1(dx^1)^2 - a_2(dx^2)^2 - a_3(dx^3)^2 - a_4(dx^4)^2 , \tag{20}$$

where a_1, a_2, a_3, a_4 depend on all spacetime variables. In order to obtain a ranking, we replace the 4 components $L_{121}, L_{131}, L_{141}, L_{122}$ and their partial derivatives by solving (4) for them and we choose an ordering $\succ$ for the P_{abce} in such a way that $P_{abc4} \succ P_{abc3} \succ P_{abc2} \succ P_{abc1}$ and then the sets of P_{abce} ordered according to $\mathcal{R}^{(W,4)}_{\succ}$ for each $e = 1, 2, 3, 4$, where $\mathcal{R}^{(W,4)}_{\succ}$ is given in Appendix A. This produces an orderly ranking and induces such a ranking amongst the *symbol variables* V_{abcd} [?, ?], where we can solve each equation for a different variable V_{abc4} corresponding to a P_{abc4}. The number in brackets indicates the corresponding Weyl-Lanczos equation or differential gauge condition given in Appendix A. Then, $\mathcal{M}_1$ is given in orthonomic form by

$$
\begin{array}{clcl}
\boxed{1} & V_{3434} & = & \frac{a_4}{a_1}V_{1331}+\frac{a_3}{a_1}V_{1441}-\frac{a_4}{a_2}V_{2332}-\frac{a_3}{a_2}V_{2442}+V_{3443} \\
\boxed{2} & V_{2434} & = & \frac{a_4}{a_1}V_{1321}-\frac{a_4}{a_2}V_{2322}+V_{2443}+V_{3442} \\
\boxed{3} & V_{2424} & = & \frac{a_4}{a_3}V_{2323}+V_{2442}-\frac{a_2a_4}{a_1a_3}V_{1331}-\frac{a_2}{a_3}V_{3443} \\
\boxed{4} & V_{2334} & = & V_{2343}+\frac{a_3}{a_2}V_{2422}+V_{3432}-\frac{a_3}{a_1}V_{1421} \\
\boxed{5} & V_{2324} & = & \frac{a_2}{a_1}V_{1431}+V_{2342}-V_{2423}-\frac{a_2}{a_3}V_{3433} \\
\boxed{6} & V_{1424} & = & \frac{a_4}{a_3}V_{1323}-\frac{a_4}{a_3}V_{1233}+V_{1442}-\frac{a_4}{a_3}V_{2331} \\
\boxed{7} & V_{1434} & = & \frac{a_4}{a_2}V_{1232}-\frac{a_4}{a_2}V_{1322}+V_{1443}+\frac{a_4}{a_2}V_{2321} \\
\boxed{8} & V_{1334} & = & V_{1343}+\frac{a_3}{a_2}V_{1422}-\frac{a_3}{a_2}V_{2421}-\frac{a_3}{a_2}V_{1242} \\
\boxed{9} & V_{1234} & = & V_{1243}+V_{1342}-V_{1432}-V_{2341}+V_{2431} \\
\boxed{10} & V_{1324} & = & V_{1243}+V_{1342}-V_{1423}+V_{2431} \\
\boxed{11} & V_{1244} & = & -\frac{a_4}{a_3}V_{2331}-V_{2441}+\frac{a_4}{a_3}V_{1332}+V_{1442}-\frac{a_4}{a_3}V_{1233} \\
\boxed{12} & V_{1344} & = & -\frac{a_4}{a_2}V_{1322}-\frac{a_4}{a_3}V_{1333}+\frac{a_4}{a_2}V_{2321}-V_{3441} \\
\boxed{13} & V_{1444} & = & -\frac{a_4}{a_2}V_{1422}-\frac{a_4}{a_3}V_{1433}+\frac{a_4}{a_2}V_{2421}+\frac{a_4}{a_3}V_{3431} \\
\boxed{14} & V_{2344} & = & -\frac{a_4}{a_1}V_{1231}+\frac{a_4}{a_1}V_{1321}-\frac{a_4}{a_2}V_{2322}-\frac{a_4}{a_3}V_{2333} \\
\boxed{15} & V_{2444} & = & -\frac{a_4}{a_1}V_{1241}+\frac{a_4}{a_1}V_{1421}-\frac{a_4}{a_2}V_{2422}-\frac{a_4}{a_3}V_{2433} \\
\boxed{16} & V_{3444} & = & -\frac{a_4}{a_1}V_{1341}+\frac{a_4}{a_1}V_{1431}+\frac{a_4}{a_2}V_{2342}-\frac{a_4}{a_2}V_{2432} \\
 & & & -\frac{a_4}{a_3}V_{3433}\,.
\end{array}
\tag{21}
$$

All 16 equations in (21) are of class 4 because all variables x^1, x^2, x^3, x^4 are multiplicative variables for each equation so that we obtain $\beta_1^{(1)} = 0, \beta_1^{(2)} = 0, \beta_1^{(3)} = 0, \beta_1^{(4)} = 16$, where the upper index (k) in $\beta_q^{(k)}$ indicates the number of multiplicative variables and the lower index the order of the system of partial differential equations. For the sum $\sum_{k=1}^{4} \beta_1^{(k)} = 16$, and because we only have 16 equations, this means that we are already using coordinates which are δ-regular. One must use δ-regular coordinates, which are coordinates which gradually maximise $\beta_q^{(n)}$, then $\beta_q^{(n-1)} + \beta_q^{(n)}$ and so on, to ensure the results obtained are intrinsic.

Then, the Cartan characters are $\alpha_1^{(1)} = 16, \alpha_1^{(2)} = 16, \alpha_1^{(3)} = 16, \alpha_1^{(4)} = 0$. They are computed according to the formula

$$
\alpha_q^{(k)} = m\binom{n+q-k-1}{q-1} - \beta_q^{(k)} \tag{22}
$$

of which details can be found in [**?**, **?**, **?**, **?**]. Now, we must verify that our symbol is involutive. Firstly, we prolong (21) by differentiating each equation in $\mathcal{R}_1$ with respect to x^1, x^2, x^3, x^4. This leads to $\mathcal{M}_2$ consisting of 64 equations. We find that we can rewrite the prolonged symbol $\mathcal{M}_2$ in such a way that each equation contains a distinct component V_{abcde} given in Appendix A so that $r(\mathcal{M}_2) = 64$. But we also have $\sum_{k=1}^{4} k \cdot \beta_1^{(k)} = 64$ which means that $\mathcal{M}_1$ is involutive, where further details on this calculation are given in [?]. In order to show that the system is formally integrable, we have to verify that the canonical projection of $\mathcal{R}_2$ from second to first order $\pi_1^2(\mathcal{R}_2)$ coincides with $\mathcal{R}_1$ itself, which means that $\pi_1^2(\mathcal{R}_2) = \mathcal{R}_1$. But our system is a linear system of PDEs and the only way an integrability condition can arise, is, if in any of the prolonged equations all the second-order partial derivatives S_{abcde} defined by $S_{abcde} = L_{abc,de}$ can be eliminated completely. This is impossible because $r(\mathcal{M}_2) = 64$ and so formal integrability follows. We conclude that we have verified by using Janet-Riquier theory that the Weyl-Lanczos equations together with the differential gauge condition form a system in involution for spacetimes with diagonal metric tensors.

If we wish to look at the symbol for a general spacetime, the calculations become more cumbersome. Therefore, we decide to look at the plane-wave limit which all spacetimes possess, and we shall analyse this system instead. A good account of the plane-wave limit of spacetimes is given in [?]. There, a part of a properly embedded null geodesic γ is taken and a corresponding procedure applied which leads to W_γ - a plane wave limit. All spacetimes can locally be expressed using a line element

$$\begin{aligned} ds^2 \quad &= 2dx^1dx^2 + a(dx^3)^2 + 2b_3dx^2dx^3 + 2b_4dx^2dx^4 - c_{33}(dx^3)^2 \\ &-2c_{34}dx^3dx^4 - c_{44}(dx^4)^2 \,, \end{aligned} \tag{23}$$

where $a, b_3, b_4, c_{33}, c_{34}, c_{44}$ are functions of all 4 coordinates. When a plane-wave limit is taken, the metric (23) becomes [?]

$$ds^2 = 2dx^1dx^2 - C_{33}(dx^3)^2 - 2C_{34}dx^3dx^4 - C_{44}(dx^4)^2 \,, \tag{24}$$

where C_{33}, C_{34} and C_{44} are arbitrary functions of x^1 only. We can determine the symbol for the metric (24) but we see that for whatever ranking we choose, the coordinates are not δ-regular. This means that we would have to prolong the system to second order and see whether we can obtain the desired result for the prolonged system. But in order to avoid this, we perform a linear coordinate transformation [?] which we choose to be

$$\begin{aligned} d\tilde{x}^1 &= a_{11}dx^1 + a_{12}dx^2 \,, \\ d\tilde{x}^2 &= a_{21}dx^1 + a_{22}dx^2 \,, \\ d\tilde{x}^3 &= dx^3 \,, \\ d\tilde{x}^4 &= dx^4 \,, \end{aligned} \tag{25}$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are arbitrary constants. To simplify matters, we first look at the special case $a_{11} = a_{12} = a_{21} = \frac{1}{\sqrt{2}}, a_{22} = -\frac{1}{\sqrt{2}}$, which produces a new, only slightly different metric line element for (24) which is

$$ds^2 = (d\tilde{x}^1)^2 - (d\tilde{x}^2)^2 - C_{33}(d\tilde{x}^3)^2 - 2C_{34}d\tilde{x}^3d\tilde{x}^4 - C_{44}(d\tilde{x}^4)^2 \,. \tag{26}$$

Even though this seems a minor transformation, the change from **characteristic** to **non-characteristic coordinates** is necessary in order to obtain intrinsic results for the Cartan characters because the new coordinates $(\tilde{x}^e)$ are δ-regular coordinates. Then, we evaluate (12) for the new line element (24) and order the P_{abcd} such that $P_{abc1} \succ P_{abc2} \succ P_{abc3} \succ P_{abc4}$ instead whereas we leave the ordering in each set P_{abce} unchanged as in $\mathcal{R}_\succ^{(W,4)}$ which leads to an orderly ranking. We can now solve each of the symbol equations for one distinct variable V_{abc1} and so obtain an orthonomic system

$$
\begin{aligned}
V_{1211} &= \frac{1}{\Delta}(V_{1222}\Delta + V_{1233}C_{44} - V_{1234}C_{34} - V_{1243}C_{34} + V_{1244}C_{33}) \\
V_{1311} &= \frac{1}{\Delta}(V_{1322}\Delta + V_{1333}C_{44} - V_{1334}C_{34} - V_{1343}C_{34} + V_{1344}C_{33}) \\
V_{1411} &= \frac{1}{\Delta}(-V_{1223}C_{34}C_{33}^{-1}\Delta - V_{1224}\Delta + V_{1333}C_{33}^{-1}C_{34}C_{44} - V_{1334}C_{44} \\
&\quad -V_{1343}C_{33}^{-1}{C_{34}}^2 + V_{1344}C_{34} + V_{1422}\Delta + V_{1433}(C_{44} - C_{33}^{-1}{C_{34}}^2)) \\
V_{1221} &= \frac{1}{\Delta}(V_{1212}\Delta - V_{1313}C_{44} + V_{1314}C_{34} + V_{1413}C_{34} - V_{1414}C_{33} \\
&\quad +V_{2323}C_{44} - V_{2324}C_{34} - V_{2423}C_{34} + V_{2424}C_{33}) \\
V_{1331} &= \frac{1}{\Delta}(V_{1212}(C_{33}{C_{34}}^3 - {C_{33}}^2C_{44}) + V_{1313}\Delta - V_{2323}{C_{34}}^2 \\
&\quad +V_{2324}C_{33}C_{34} - V_{2332}\Delta + V_{2423}C_{33}C_{34} - V_{2424}{C_{33}}^2) \\
V_{2321} &= \frac{1}{\Delta}(V_{1223}\Delta - V_{1232}\Delta + V_{1322}\Delta + V_{1333}C_{44} - V_{1334}C_{34} - V_{1433}C_{34} \\
&\quad +V_{1434}C_{33}) \\
V_{2331} &= \frac{1}{\Delta}(V_{1222}(C_{33} - {C_{33}}^2C_{44}) - V_{1233}\Delta - V_{1332}\Delta + V_{1423}C_{33}C_{34} \\
&\quad -V_{1424}{C_{33}}^2 + V_{1323}(C_{33}C_{44} - 2{C_{34}}^2) + V_{1324}C_{33}C_{34}) \\
V_{2421} &= \frac{1}{\Delta}(V_{1223}(C_{34}C_{44} - C_{33}^{-1}{C_{34}}^3) - V_{1242}\Delta + V_{1333}C_{33}^{-1}C_{34}C_{44} \\
&\quad -V_{1334}C_{44} + V_{1343}C_{33}^{-1}\Delta + V_{1422}\Delta - V_{1433}C_{33}^{-1}{C_{34}}^2 + V_{1434}C_{34}) \\
V_{1231} &= \frac{1}{\Delta}(V_{1213}\Delta + V_{1312}\Delta - V_{2322}\Delta + V_{2343}C_{34} - V_{2344}C_{33} - V_{2433}C_{34} \\
&\quad +V_{2434}C_{33}) \\
V_{1321} &= \frac{1}{\Delta}(V_{1213}\Delta + V_{1312}\Delta + V_{2333}C_{44} - V_{2334}C_{34} - V_{2433}C_{34} + V_{2434}C_{33}) \\
V_{1241} &= \frac{1}{\Delta}(V_{1214}\Delta + V_{1412}\Delta + V_{2343}C_{44} - V_{2344}C_{34} - V_{2422}\Delta \\
&\quad +V_{2434}(C_{34} - C_{44})) \\
V_{1421} &= \frac{1}{\Delta}(V_{1213}(C_{34}C_{44} - C_{33}^{-1}{C_{34}}^3) + V_{1412}\Delta + V_{2333}C_{33}^{-1}C_{34}C_{44} \\
&\quad -V_{2334}C_{44} + V_{2343}C_{33}^{-1}\Delta - V_{2433}C_{33}^{-1}{C_{34}}^2 + V_{2434}C_{34}) \\
V_{1341} &= \frac{1}{\Delta}(-V_{1212}C_{34}\Delta - V_{1314}\Delta - V_{2323}C_{33}C_{44} + V_{2324}{C_{34}}^2 + V_{2423}C_{33} \\
&\quad C_{44} - V_{2424}C_{33}C_{34} - V_{2432}\Delta) \\
V_{1431} &= \frac{1}{\Delta}(-V_{1212}C_{34}\Delta + V_{1413}\Delta - V_{2323}C_{34}C_{44} + V_{2324}C_{33}C_{44} - V_{2342}\Delta \\
&\quad +V_{2424}{C_{34}}^2 - V_{2424}C_{33}C_{34}) \\
V_{2341} &= \frac{1}{\Delta}(-V_{1222}C_{34}\Delta - V_{1234}\Delta - V_{1323}C_{34}C_{44} + V_{1324}C_{33}C_{44} \\
&\quad +V_{1423}C_{33}C_{44} - V_{1424}C_{33}C_{34} - V_{1432}\Delta) \\
V_{2431} &= \frac{1}{\Delta}(-V_{1222}C_{34}\Delta - V_{1243}\Delta - V_{1323}C_{34}C_{44} + V_{1324}C_{33}C_{44} - V_{1342}\Delta \\
&\quad +V_{1423}C_{33}C_{44} - V_{1424}C_{33}C_{34})\,, \qquad (27)
\end{aligned}
$$

where $\Delta = C_{33}C_{44} - {C_{34}}^2$. This system is composed of 16 equations all of class 4 which leads to $\beta_1^{(1)} = \beta_1^{(2)} = \beta_1^{(3)} = 0, \beta_1^{(4)} = 16$ and therefore to the Cartan characters $\alpha_1^{(1)} = \alpha_1^{(2)} = \alpha_1^{(3)} = 16$ and $\alpha_1^{(4)} = 0$. This result is intrinsic because our coordinates are now δ-regular.

Next, we prolong each of the 16 equations and obtain 64 equations which can be modified so that each of the corresponding symbol equations contains a distinct variable $V_{abc11}, V_{abc22}, V_{abc33}, V_{abc44}$ or V_{abc34} and therefore $r(\mathcal{M}_2) = 64$. This agrees with $\sum_{k=1}^{4} k \cdot \beta_1^{(k)} = 64$ so that $\mathcal{M}_1$ is involutive. No integrability conditions can occur for the same reasons as for the system (21) above and the system is in involution with Cartan characters $(16, 16, 16, 0)$.

We obtain the same result for the plane wave limit of any spacetime as for the for the case of spacetimes with diagonal metric and we conjecture that the same system on spacetimes with arbitrary analytic metric will be in involution with Cartan characters $(16, 16, 16, 0)$.

III. The Lanczos Wave Equation in 4 Dimensions

In this section, we look at the Lanczos wave equation at first as an EDS and then as a system of PDEs. We again determine the Cartan characters and show that it consists of a system in involution using both theories, a result which can be derived from the scalar wave equation directly.

A. The Lanczos Wave Equation as an EDS

We can also describe the Lanczos tensor wave equation in terms of an EDS on a jet bundle $\mathcal{J}^2(\mathbb{R}^4, \mathbb{R}^{16})$ with formal dimension $N = 244$ for which we choose the local coordinates $(x^e, L_{abc}, P_{abcd}, S_{abcde})$ composed by 4 spacetime coordinates x^e, 16 L_{abc}, 64 P_{abcd} and 256 S_{abcde}. Here, again S_{abcde} are the variables corresponding to the second-order partial derivatives of the L_{abc} when projected onto our spacetime manifold.

First, we write the Lanczos wave equation (8) in solved form and denote its components by W_{abc}. In a local coordinate frame W_{abc} is then given by

$$\begin{aligned} W_{abc} \;=\; & g^{sm}[S_{abcms} - \Gamma^n_{am,s}L_{nbc} - \Gamma^n_{bm,s}L_{anc} - \Gamma^n_{cm,s}L_{abn} - \Gamma^n_{as}P_{nbcm} \\ & + \Gamma^n_{as}\Gamma^k_{nm}L_{kbc} + \Gamma^n_{as}\Gamma^k_{bm}L_{nkc} + \Gamma^n_{as}\Gamma^k_{cm}L_{nbk} - \Gamma^n_{bs}P_{ancm} + \Gamma^n_{bs}\Gamma^k_{am} \\ & L_{knc} + \Gamma^n_{bs}\Gamma^k_{nm}L_{akc} + \Gamma^n_{bs}\Gamma^k_{cm}L_{ank} - \Gamma^n_{cs}P_{abnm} + \Gamma^n_{cs}\Gamma^k_{am}L_{kbn} \\ & + \Gamma^n_{cs}\Gamma^k_{bm}L_{akn} + \Gamma^n_{cs}\Gamma^k_{nm}L_{abk} - \Gamma^n_{ms}P_{abcn} + \Gamma^n_{ms}\Gamma^k_{an}L_{kbc} \\ & + \Gamma^n_{ms}\Gamma^k_{bn}L_{akc} + \Gamma^n_{ms}\Gamma^k_{cn}L_{abk}] + 2R_c{}^sL_{abs} - R_a{}^sL_{bcs} - R_b{}^sL_{cas} \\ & - g_{ac}R^{ls}L_{lbs} + g_{bc}R^{ls}L_{las} - \frac{1}{2}RL_{abc} - J_{abc} \,. \end{aligned} \tag{28}$$

In addition to the exterior derivatives of (28), we need to add two sets of contact conditions, K_{abc} and K_{abcd}, when we write (28) as an EDS. Altogether, we obtain the Pfaffian system

$$\begin{aligned} dW_{abc} &= dW_{abc} \\ K_{abc} &= dL_{abc} - P_{abce}dx^e \\ K_{abcd} &= dP_{abcd} - S_{abcde}dx^e \,, \end{aligned} \tag{29}$$

where dW_{abc} are the exterior derivatives of the components of the wave equation in solved form, which are locally given by

$$\begin{aligned} dW_{abc} \;=\; & [\Box L_{abc} + 2R_c{}^sL_{abs} - R_a{}^sL_{bcs} - R_b{}^sL_{cas} - g_{ac}R^{ls}L_{lbs} + g_{bc}R^{ls}L_{las} \\ & - \frac{1}{2}RL_{abc} - J_{abc}]_{,e}dx^e + g^{sm}[dS_{abcms} - \Gamma^n_{am,s}dL_{nbc} - \Gamma^n_{bm,s}dL_{anc} \end{aligned}$$

$$-\Gamma^n_{cm,s}dL_{abn} - \Gamma^n_{as}dP_{nbcm} + \Gamma^n_{as}\Gamma^k_{nm}dL_{kbc} + \Gamma^n_{as}\Gamma^k_{bm}dL_{nkc}$$
$$+\Gamma^n_{as}\Gamma^k_{cm}dL_{nbk} - \Gamma^n_{bs}dP_{ancm} + \Gamma^n_{bs}\Gamma^k_{am}dL_{knc} + \Gamma^n_{bs}\Gamma^k_{nm}dL_{akc}$$
$$+\Gamma^n_{bs}\Gamma^k_{cm}dL_{ank} - \Gamma^n_{cs}dP_{abnm} + \Gamma^n_{cs}\Gamma^k_{am}dL_{kbn} + \Gamma^n_{cs}\Gamma^k_{bm}dL_{akn}$$
$$+\Gamma^n_{cs}\Gamma^k_{nm}dL_{abk} - \Gamma^n_{ms}dP_{abcn} + \Gamma^n_{ms}\Gamma^k_{an}dL_{kbc} + \Gamma^n_{ms}\Gamma^k_{bn}dL_{akc}$$
$$+\Gamma^n_{ms}\Gamma^k_{cn}dL_{abk}] + 2R^{\ s}_{c}dL_{abs} - R^{\ s}_{a}dL_{bcs} - R^{\ s}_{b}dL_{cas}$$
$$-g_{ac}R^{ls}dL_{lbs} + g_{bc}R^{ls}dL_{las} - \frac{1}{2}RdL_{abc} \,. \tag{30}$$

A Vessiot vector field for (29) is of the form

$$V = V^e\frac{\partial}{\partial x^e} + V^e P_{abce}\frac{\partial}{\partial L_{abc}} + V^e S_{abcde}\frac{\partial}{\partial P_{abcd}} + V_{abcde}\frac{\partial}{\partial S_{abcde}} \,, \tag{31}$$

where 16 of the 160 V_{abcde} are determined by requiring that $dW_{abc}(V) = 0$. When we apply the 2-forms dK_{abc}, dK_{abcd} to the two Vessiot vector fields V^1 and V^2, we can see that $dx^e \wedge dP_{abce}(V^1, V^2)$ vanishes identically. Therefore, we only have to examine the second set $dK_{abcd} = -dx^e \wedge dS_{abcde}$ when we form our integral elements of dimensions greater than one.

When we start to compute the Cartan characters of (29), we obtain the values $s_0 = s_0' = 96$ again by counting the independent 1-forms in (29) for the full and for the reduced system respectively. These 1-forms consist of 16 exterior derivatives of the Lanczos wave equation, 16 first-order contact conditions K_{abc} and 64 second-order contact conditions K_{abcd} totalling 96 independent 1-forms. If we pull the system back onto the submanifold defined by $W_{abc} = 0$, we obtain $s_0 = s_0' = 80$.

The associated system of (29) can again be determined and it is given by

$$\{dW_{abc}, K_{abc}, K_{abcd}, \omega^e, dS_{abcde}\} \,,$$

where 16 of the totally 160 dS_{abcde} can be expressed by solving dW_{abc} for them so that we only add the 144 remaining dS_{abcde} for which the dW_{abc} are not solved for. We see that $\dim(\mathcal{A}(\mathcal{P})) = 244 = c$ which means that $\dim(\bar{\mathcal{D}}) = 0$ so that again, no Cauchy characteristics can exist because for an $Y \in Char(\mathcal{P})$

$$Y \rfloor dK_{abcd} = Y^e dS_{abcde} - Y_{abcde}dx^e$$

has to be such that its RHS can be expressed by means of a non-trivial linear combination of the form

$$\lambda^{a'b'c'}K_{a'b'c'} + \lambda^{a'b'c'd'}K_{a'b'c'd'} + \lambda^{mns}dW_{mns} \,, \tag{32}$$

which is impossible.

Again, we wish to determine the reduced Cartan characters using the tableau matrices $A^{\alpha}_{\lambda i}$ and examine whether the torsion terms B^{α}_{ij} of (29) can really be absorbed. Therefore, we must complete $(dW_{abc}, K_{abc}, K_{abcd}, \omega^i)$ to a coframe on our formally 244-dimensional jet-bundle. In (29) we have 96 one-forms to which we add the 4 one-forms $\omega^1 = dx^1$, $\omega^2 = dx^2$, $\omega^3 = dx^3$, $\omega^4 = dx^4$ again composing the independence condition $\Omega \neq 0$ so that we need to add a further 144 forms π^Λ in order to obtain a complete coframe of $N = 244$ one-forms. Again, Λ is a collective index again subject to Einstein's summation convention, this time replacing certain sets of indices $abcd$ in dS_{abcde}. In order to obtain intrinsic values, we choose to replace all components dS_{abcde}, where either d or e are $= 4$, first, and so on based on x^4

being our first coordinate, then x^3, then x^2 and x^1 last. The correspondence of $\pi^\Lambda \leftrightarrow dS_{abcde}$ is given in detail in Appendix B. Note that we did not replace the 16 dS_{abc11} as this would lead to an excess of 1-forms in our coframe but we solve each dW_{abc} for a distinct dS_{abc11}.

Because $dK_{abc} = 0 \pmod{(\mathcal{P})}$, we only need to consider dK_{abcd} when computing the tableau matrix and the reduced Cartan characters. We can now write the 64 dK_{abcd} as

$$d\theta^\alpha \equiv A^\alpha_{\Lambda 2}\pi^\Lambda\wedge\omega^2 + A^\alpha_{\Lambda 3}\pi^\Lambda\wedge\omega^3 + A^\alpha_{\Lambda 4}\pi^\Lambda\wedge\omega^4 - dS_{abc11}\wedge\omega^1 + B^\alpha_{ij}\omega^i\wedge\omega^j . \quad (33)$$

Here, we assumed that we can express all 16 dS_{abc11} by means of a distinct linear combination of the dW_{abc}, K_{abc}, K_{abcd} and the ω^e. From this we find that the only non-vanishing tableau matrix components $A^\alpha_{\Lambda i}$ for $i \neq 1$ are given by $A^\alpha_{\Lambda i} = -1$ when the indices α and Λ correspond to the same set of indices $abcd$ as given in Appendix B and $A^\alpha_{\Lambda i} = 0$ otherwise. This leads to the tableau matrix

$$\begin{pmatrix} A^\alpha_{\Lambda\,4} \\ A^\alpha_{\Lambda\,3} \\ A^\alpha_{\Lambda\,2} \\ A^\alpha_{\Lambda\,1} \end{pmatrix}$$

which is a 160x144-matrix of the form

$$M := \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \\ X_1 & X_2 & X_3 \end{pmatrix} ,$$

where X_1, X_2, X_3 stand for $A^\alpha_{\Lambda 1}$ and A is a 64x64-matrix of the form $-\mathbb{I}_{64}$, B a 48x48-matrix $-\mathbb{I}_{48}$ and C a 32x32-matrix $-\mathbb{I}_{32}$. We see that even without the 16 rows for the entries X the maximal rank $r(M) = 144$ is obtained. This immediately leads to $s'_1 = 64$,$s'_2 = 48$, $s'_3 = 32$ and $s'_4 = 0$ meaning that the set of reduced Cartan characters $(s'_0, s'_1, s'_2, s'_3, s'_4)$is $(96, 64, 48, 32, 0)$.

As a last step, we again give an Ansatz for a transformation Φ by means of which the torsion terms B^α_{ij} can be absorbed. We perform a transformation ϕ such that

$$0 = B^\alpha_{i1} - A^\alpha_{\Lambda 1}p^\Lambda_i, \quad i \neq 1 \;, \quad (34)$$

and we set $p^\Lambda_1 := 1$ because all B^α_{ij} for both $i, j \neq 1$ vanish identically. This leads to $\tilde{B}^\alpha_{i1} = 0$ for any such solution for the p^Λ_i. Then, the torsion terms are absorbed and the system is in involution. Again, we do not carry out this longish calculation by hand but refer to results in [**?**] confirming that the Lanczos wave equation consists of a system in involution with Cartan characters $(96, 64, 48, 32, 0)$. For a number of spacetimes such as Kasner, Gödel,Schwarzschild, conformally flat spacetimes, this result was shown using the REDUCE code given in [**?**].

B. The Lanczos Wave Equation as a System of PDEs

The Lanczos wave equation constitutes of 16 second-order equations for the 16 independent Lanczos components, where we again imposed (3) and (4). We can determine the symbol of the wave equation and from this obtain the Cartan characters. We know that the wave equation has the form (8). We denote the system formed by these 16 components of the wave equation by $\mathcal{R}_{2,wave}$, where the index "2" refers to the order of the system. Due to the definition of the symbol, only the highest-order derivatives matter, which are second-order derivatives here. Therefore, the only terms contributing to the symbol $\mathcal{M}_{2,wave}$ are parts of the $\Box L_{abc}$-terms, namely all the $g^{sm}\partial_s\partial_m L_{abc}$. The 16 equations for the symbol for an arbitrary spacetime then look like

$$\begin{aligned} V_{abc44} &= \frac{1}{g^{44}}(g^{11}V_{abc11} + g^{22}V_{abc22} + g^{33}V_{abc33} + 2g^{12}V_{abc12} + 2g^{13}V_{abc13} \\ &\quad +2g^{14}V_{abc14} + 2g^{23}V_{abc23} + 2g^{24}V_{abc24} + 2g^{34}V_{abc34}) \,. \end{aligned} \tag{35}$$

We can easily see that by ordering the S_{abcde} in such a way that $S_{abc44} \succ S_{abc34} \succ S_{abc33} \succ S_{abc24} \succ S_{abc23} \succ S_{abc22} \succ S_{abc14} \succ S_{abc13} \succ S_{abc12} \succ S_{abc11}$ based on $x^4 \succ x^3 \succ x^2 \succ x^1$ and by choosing for each set of S_{abcij} the ordering $\mathcal{R}_{\succ}^{(wave,4)}$ given in Appendix B, an orderly ranking is achieved.

Because all equations of form (35) are of class 4, it means that $\beta_2^{(4)} = 16$ is the maximal value for $\beta_2^{(4)}$. Because all $\beta_2^{(k)} = 0$ for $k < 4$, the decreasing sums of the $\beta_2^{(k)}$ are automatically 16, which is the maximal value, so that the system is given in δ-regular coordinates. Therefore, the following values for $\beta_2^{(1)} = 0, \beta_2^{(2)} = 0, \beta_2^{(3)} = 0, \beta_2^{(4)} = 16$ lead to the intrinsic results for the Cartan characters which are $\alpha_2^{(1)} = 64, \alpha_2^{(2)} = 48, \alpha_2^{(3)} = 32, \alpha_2^{(4)} = 0$.

In order to see whether the system is in involution, we determine $\mathcal{M}_{3,wave}$. Each component of the wave equation prolonged to third order contributes to the symbol $\mathcal{M}_{3,wave}$ with

$$\begin{aligned} 0 &= g^{11}V_{abc111} + g^{22}V_{abc122} + g^{33}V_{abc133} + g^{44}V_{abc144} + 2g^{12}V_{abc112} \\ &\quad +2g^{13}V_{abc113} + 2g^{14}V_{abc114} + 2g^{23}V_{abc123} + 2g^{24}V_{abc124} + 2g^{34}V_{abc134} \\ 0 &= g^{11}V_{abc112} + g^{22}V_{abc222} + g^{33}V_{abc233} + g^{44}V_{abc244} + 2g^{12}V_{abc122} \\ &\quad +2g^{13}V_{abc123} + 2g^{14}V_{abc124} + 2g^{23}V_{abc223} + 2g^{24}V_{abc224} + 2g^{34}V_{abc234} \\ 0 &= g^{11}V_{abc113} + g^{22}V_{abc223} + g^{33}V_{abc333} + g^{44}V_{abc344} + 2g^{12}V_{abc123} \\ &\quad +2g^{13}V_{abc133} + 2g^{14}V_{abc134} + 2g^{23}V_{abc233} + 2g^{24}V_{abc234} + 2g^{34}V_{abc334} \\ 0 &= g^{11}V_{abc114} + g^{22}V_{abc224} + g^{33}V_{abc334} + g^{44}V_{abc444} + 2g^{12}V_{abc124} + 2g^{13} \\ &\quad V_{abc134} + 2g^{14}V_{abc144} + 2g^{23}V_{abc234} + 2g^{24}V_{abc244} + 2g^{34}V_{abc344} \,, \end{aligned} \tag{36}$$

where 320 distinct symbol variables V_{abcdef} can occur corresponding to the 320 distinct third order partial derivatives which can occur. This leads to a 64x320-symbol matrix

$$\begin{pmatrix} S & 0 & \cdots & 0 \\ 0 & S & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S \end{pmatrix},$$

where each of the 16 matrices S have the form of the 4x20-matrix

$$\left(\begin{array}{cccccccccccccccccccc} g^{11} & 2g^{12} & 2g^{13} & 2g^{14} & g^{22} & 2g^{23} & 2g^{24} & g^{33} & 2g^{34} & g^{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g^{11} & 0 & 0 & 2g^{12} & 2g^{13} & 2g^{14} & 0 & 0 & 0 & g^{22} & 2g^{23} & 2g^{24} & g^{33} & 2g^{34} & g^{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & g^{11} & 0 & 0 & 2g^{12} & 0 & 2g^{13} & 2g^{14} & 0 & 0 & g^{22} & 0 & 2g^{23} & 2g^{24} & 0 & g^{33} & 2g^{34} & g^{44} & 0 \\ 0 & 0 & 0 & g^{11} & 0 & 0 & 2g^{12} & 0 & 2g^{13} & 2g^{14} & 0 & 0 & g^{22} & 0 & 2g^{23} & 2g^{24} & 0 & g^{33} & 2g^{34} & g^{44} \end{array}\right).$$

From this we can easily deduce that the rank of S is always 4 independently of the choice of the metric tensor so that $r(\mathcal{M}_{3,wave}) = 4 \cdot 16 = 64$ holds. But this is equal to $\sum_{k=1}^{4} k \cdot \beta_2^{(k)} = 4 \cdot 16 = 64$ so that $\mathcal{M}_{2,wave}$ is involutive. In order to show that $\mathcal{R}_{2,wave}$ is formally integrable, we need to show that for the canonical projection $\pi_2^3(\mathcal{R}_{3,wave}) = \mathcal{R}_{2,wave}$. But again, our system is linear, and the only way an integrability condition can occur for an involutive symbol derived from a system of linear PDEs, is, if one of the equations in $\mathcal{R}_{3,wave}$ can be modified in such a way that no third-order partial derivatives occur. Because $r(\mathcal{M}_{3,wave}) = 64$ is maximal, this is not possible and the Lanczos tensor wave equation forms a system in involution with Cartan characters $(64, 48, 32, 0)$.

We can also derive this result directly from the *scalar wave equation*

$$\Box\Psi = 0\,, \tag{37}$$

where Ψ is our scalar component depending on x^1, x^2, x^3, x^4. For the scalar wave equation we use a formal manifold $\mathcal{M}$ with $N = 19$ formal dimensions of which a local basis is given by the 4 x^e, 1 dependent variable Ψ, 4 P_e and 10 S_{ef} on our jet bundle $\mathcal{J}^2(\mathbb{R}^4, \mathbb{R})$. When projected onto the 4-dimensional spacetime manifold, P_e are the first-order and S_{ef} are the second-order partial derivatives of Ψ. Trivially, the symbol of a single equation is always involutive, no matter what the ranking we chose, is, so that we obtain $\beta_2^{(1)} = \beta_2^{(2)} = \beta_2^{(3)} = 0$ and $\beta_2^{(4)} = 1$ based on the fact that a single equation is always of class n. We then obtain the characters

$$\alpha_2^{(1)} = 4\,, \alpha_2^{(2)} = 3\,, \alpha_2^{(3)} = 2\,, \alpha_2^{(4)} = 0\,.$$

We further find that $r(\mathcal{M}_3) = 4$ and therefore $\dim(\mathcal{M}_3) = 16$ whereas $r(\mathcal{R}_3) = 5$ and $\dim(\mathcal{R}_3) = 30$. We also find that $\dim(\mathcal{R}_2) = 14$ and because the symbol is linear it

$$\dim(\mathcal{R}_2^{(1)}) = \dim(\mathcal{R}_3) - \dim(\mathcal{M}_3) = 30 - 16 = 14\,.$$

From this we conclude that $\dim(\mathcal{R}_2) = \dim(\mathcal{R}_2^{(1)}) = 14$ and, because the system is linear, no integrability conditions can occur and the scalar wave equation (37) consists of a system in involution with characters $(4, 3, 2, 0)$. Note that in the case of the Lanczos wave equation, we are dealing with 16 such equations because we have 16 independent components L_{abc} and find the correspondence between the two sets of characters is given by

$$16 \cdot (4, 3, 2, 0) = (64, 48, 32, 0)\,.$$

Lastly, we wish to compare the characters of the Weyl-Lanczos equations with those of the Lanczos wave equation. Therefore, we need to prolongate the Weyl-Lanczos equations once to second order. But for an involutive system the characters of the symbol $\mathcal{M}_{q+r}$ of the prolonged system together with the $\beta_{q+r}^{(k)}$ are directly determined by [**?**, **?**]

$$\begin{aligned} \alpha_{q+r}^{k} &= \sum_{i=k}^{n} \binom{r+i-k-1}{r-1} \alpha_q^{(i)}\,, \\ \beta_{q+r}^{k} &= \sum_{i=k}^{n} \binom{r+i-k-1}{r-1} \beta_q^{(i)}\,, \end{aligned} \tag{38}$$

where in our case $n = 4, m = 16, q = 1, r = 1$, which leads to the result (48,32,16,0) for $(\alpha_2^{(1)}, \alpha_2^{(2)}, \alpha_2^{(3)}, \alpha_2^{(4)})$ for the prolonged Weyl-Lanczos equations. This shows that, based on the $\alpha_2^{(k)}$ for both systems, the Weyl-Lanczos equations are more restrictive than the Lanczos wave equation with characters (64,48,32,0). The general solution of the Weyl-Lanczos equations only depends on 16 arbitrary functions of 3 variables whereas the general solution for the Lanczos wave equations contains 32 arbitrary functions of 3 variables.

Conclusion

We obtained the Cartan characters $(s_0, s_1, s_2, s_3, s_4)$ for the Weyl-Lanczos equations given as a Pfaffian system in involution which are (32,16,16,16,0) and (16,16,16,16,0) when pulled back onto the submanifold where the Weyl-Lanczos equations themselves vanish identically. These results were obtained assisted by REDUCE codes based on the EDS package. The Cartan characters could be obtained for a diagonalized spacetime and for the plane wave limit taken when using Janet-Riquier theory and they are (16,16,16,0).

For the Lanczos wave equation, we showed that it also consists of a Pfaffian system in involution and that its Cartan characters are given by (96,64,48,32,0) or by (80,64,48,32,0) when pulled back. We related the Cartan characters and involutiveness of the Lanczos wave equation to the corresponding results for the scalar wave equation. We also found that the Lanczos wave equation is less restrictive than the Weyl-Lanczos equations allowing for 32 arbitrary functions of 3 variables as opposed to only 16.

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Appendix A: The Weyl-Lanczos Equations

Here, we give the expressions for the α_{abcde} and the γ_{abcde} occurring in the EDS for the Weyl-Lanczos equations given by (13). The quantity α_{abcde} is determined through

$$\begin{aligned}\alpha_{abcde} = {} & \Gamma^n_{ad,e}(L_{nbc} + L_{ncb}) + \Gamma^n_{bc,e}(L_{nad} + L_{nda}) - \Gamma^n_{ac,e}(L_{nbd} + L_{ndb}) \\ & -\Gamma^n_{bd,e}(L_{nac} + L_{nca})\,. \end{aligned} \tag{A.1}$$

The quantity γ_{abcde} is given by

$$\begin{aligned}\gamma_{abcde} = {} & g_{bc}[P_{nads}g^{ns}_{\ \ ,e} - L_{mad}(\Gamma^m_{ns}g^{ns})_{,e} - L_{nmd}(\Gamma^m_{as}g^{ns})_{\,,e} - L_{nam}(\Gamma^m_{ds}g^{ns})_{,e}] \\ & +g_{ad}[P_{nbcs}g^{ns}_{\ \ ,e} - L_{mbc}(\Gamma^m_{ns}g^{ns})_{,e} - L_{nmc}(\Gamma^m_{bs}g^{ns})_{\,,e} - L_{nbm}(\Gamma^m_{cs}g^{ns})_{,e}] \\ & -g_{bd}[P_{nacs}g^{ns}_{\ \ ,e} - L_{mac}(\Gamma^m_{ns}g^{ns})_{,e} - L_{nmc}(\Gamma^m_{as}g^{ns})_{\,,e} - L_{nam}(\Gamma^m_{cs}g^{ns})_{,e}] \\ & -g_{ac}[P_{nbds}g^{ns}_{\ \ ,e} - L_{mbd}(\Gamma^m_{ns}g^{ns})_{,e} - L_{nmd}(\Gamma^m_{bs}g^{ns})_{\,,e} - L_{nbm}(\Gamma^m_{ds}g^{ns})_{,e}] \\ & +g_{bc,e}[P_{nads}g^{ns} - L_{mad}(\Gamma^m_{ns}g^{ns}) - L_{nmd}(\Gamma^m_{as}g^{ns}) - L_{nam}(\Gamma^m_{ds}g^{ns})] \\ & +g_{ad,e}[P_{nbcs}g^{ns} - L_{mbc}(\Gamma^m_{ns}g^{ns}) - L_{nmc}(\Gamma^m_{bs}g^{ns}) - L_{nbm}(\Gamma^m_{cs}g^{ns})] \\ & -g_{bd,e}[P_{nacs}g^{ns} - L_{mac}(\Gamma^m_{ns}g^{ns}) - L_{nmc}(\Gamma^m_{as}g^{ns}) - L_{nam}(\Gamma^m_{cs}g^{ns})] \\ & -g_{ac,e}[P_{nbds}g^{ns} - L_{mbd}(\Gamma^m_{ns}g^{ns}) - L_{nmd}(\Gamma^m_{bs}g^{ns}) \\ & -L_{nbm}(\Gamma^m_{ds}g^{ns})]\,. \end{aligned} \tag{A.2}$$

When introducing the collective index Λ for the additional coframe elements π^Λ, we ordered the L_{abc} according to

$$L_{133} < L_{144} < L_{123} < L_{132} < L_{124} < L_{142} < L_{134} < L_{143}$$

$$< L_{232} < L_{233} < L_{242} < L_{244} < L_{234} < L_{243} < L_{343} < L_{344}$$

from which we deduce the ordering for the $\pi^\Lambda \leftrightarrow dP_{abce}$ in such a way that the dP_{abce} are arranged like the L_{abc} above. In this way the π^Λ are labelled according to the correspondence

$$\begin{array}{lll} \pi^1 \leftrightarrow dP_{1331} & , \cdots , & \pi^{16} \leftrightarrow dP_{3441} \, , \\ \pi^{17} \leftrightarrow dP_{1332} & , \cdots , & \pi^{32} \leftrightarrow dP_{3442} \, , \\ \pi^{33} \leftrightarrow dP_{1333} & , \cdots , & \pi^{48} \leftrightarrow dP_{abc3} \, . \end{array}$$

When expressing the Weyl-Lanczos equations as a system of PDEs, we used the standard orderly ranking (if not stated otherwise explicitely) which is based on the ordering $P_{abc4} \succ P_{abc3} \succ P_{abc2} \succ P_{abc1}$. Then, amongst each set of the P_{abci} we order them according to

$$P_{344i} \succ P_{343i} \succ P_{243i} \succ P_{234i} \succ P_{244i} \succ P_{242i} \succ P_{233i} \succ P_{232i}$$

$$\succ P_{143i} \succ P_{134i} \succ P_{142i} \succ P_{124i} \succ P_{132i} \succ P_{123i} \succ P_{144i} \succ P_{133i} \, .$$

This leads to an orderly ranking for the P_{abcd} for the Weyl-Lanczos equations denoted by $\mathcal{R}_\succ^{(W,4)}$.

Sometimes, we also use the following labelling for the Weyl-Lanczos equations and the differential gauge conditions:

$$\begin{array}{llll} \boxed{1} \leftrightarrow f_{1212} \, , & \boxed{2} \leftrightarrow f_{1213} \, , & \boxed{3} \leftrightarrow f_{1313} \, , & \boxed{4} \leftrightarrow f_{1214} \, , \\ \boxed{5} \leftrightarrow f_{1314} \, , & \boxed{6} \leftrightarrow f_{1323} \, , & \boxed{7} \leftrightarrow f_{1223} \, , & \boxed{8} \leftrightarrow f_{1224} \, , \\ \boxed{9} \leftrightarrow f_{1234} \, , & \boxed{10} \leftrightarrow f_{1324} \, , & \boxed{11} \leftrightarrow L_{12}{}^{s}{}_{;s} \, , & \boxed{12} \leftrightarrow L_{13}{}^{s}{}_{;s} \, , \\ \boxed{13} \leftrightarrow L_{14}{}^{s}{}_{;s} \, , & \boxed{14} \leftrightarrow L_{23}{}^{s}{}_{;s} \, , & \boxed{15} \leftrightarrow L_{24}{}^{s}{}_{;s} \, , & \boxed{16} \leftrightarrow L_{34}{}^{s}{}_{;s} \, . \end{array}$$

Lastly, we give independent symbol components V_{abcde} for each of the 64 equations for the prolonged symbol $\mathcal{M}_2$ for the Weyl-Lanczos equations which are:

	i = 1	i = 2	i = 3	i = 4
$\partial_i f_{1212}$	V_{14411}	V_{23322}	V_{34433}	V_{34344}
$\partial_i f_{1213}$	V_{13211}	V_{34422}	V_{24433}	V_{24344}
$\partial_i f_{1313}$	V_{13311}	V_{24422}	V_{23233}	V_{24244}
$\partial_i f_{1214}$	V_{14211}	V_{34322}	V_{23433}	V_{23344}
$\partial_i f_{1314}$	V_{14311}	V_{23422}	V_{24233}	V_{23244}
$\partial_i f_{1323}$	V_{23311}	V_{14422}	V_{13233}	V_{14244}
$\partial_i f_{1223}$	V_{23211}	V_{12322}	V_{14433}	V_{14344}
$\partial_i f_{1224}$	V_{24211}	V_{12422}	V_{13433}	V_{13344}
$\partial_i f_{1234}$	V_{23411}	V_{14322}	V_{12433}	V_{12344}
$\partial_i f_{1324}$	V_{24311}	V_{13422}	V_{14233}	V_{13244}
$\partial_i L_{12}{}^{s}{}_{;s}$	V_{24411}	V_{13322}	V_{12333}	V_{12444}
$\partial_i L_{13}{}^{s}{}_{;s}$	V_{34411}	V_{13222}	V_{13333}	V_{13444}
$\partial_i L_{14}{}^{s}{}_{;s}$	V_{34311}	V_{14222}	V_{14333}	V_{14444}
$\partial_i L_{23}{}^{s}{}_{;s}$	V_{12311}	V_{23222}	V_{23333}	V_{23444}
$\partial_i L_{24}{}^{s}{}_{;s}$	V_{12411}	V_{24222}	V_{24333}	V_{24444}
$\partial_i L_{34}{}^{s}{}_{;s}$	V_{13411}	V_{24322}	V_{34333}	V_{34444}

.

Based on this we can show that $r(\mathcal{M}_2) = 64$, where the detailed calculation is given in [**?**].

Appendix B: The Lanczos Wave Equation

We also need to specify the correspondence between the π^Λ and the dS_{abcde} for the Lanczos wave equation. For both i, j fixed we assume that the dS_{abcij} are ordered according to the ordering of the L_{abc} in Appendix A. For the first 64 π^Λ, where we have used the fact that $x^4 \succ x^3 \succ x^2 \succ x^1$ for our independent variables in order to obtain intrinsic results, we obtain the labelling (for the pairs of indices $ij = 44, 34, 24, 14$)

$$\begin{array}{lcl}
\pi^1 \leftrightarrow dS_{13344} & , \cdots , & \pi^{16} \leftrightarrow dS_{34444} \, , \\
\pi^7 \leftrightarrow dS_{13334} & , \cdots , & \pi^{32} \leftrightarrow dS_{34434} \, , \\
\pi^{33} \leftrightarrow dS_{13324} & , \cdots , & \pi^{48} \leftrightarrow dS_{34424} \, , \\
\pi^{49} \leftrightarrow dS_{13314} & , \cdots , & \pi^{64} \leftrightarrow dS_{34414} \, .
\end{array}$$

Then, the next 48 π^Λ are given by (for $ij = 33, 23, 13$)

$$\begin{array}{lcl}
\pi^{65} \leftrightarrow dS_{13333} & , \cdots , & \pi^{80} \leftrightarrow dS_{34433} \, , \\
\pi^{81} \leftrightarrow dS_{13323} & , \cdots , & \pi^{96} \leftrightarrow dS_{34423} \, , \\
\pi^{97} \leftrightarrow dS_{13313} & , \cdots , & \pi^{112} \leftrightarrow dS_{34413} \, .
\end{array}$$

The last 32 π^Λ are denoted as follows (for $ij = 22, 12$)

$$\begin{array}{lcl}
\pi^{113} \leftrightarrow dS_{13322} & , \cdots , & \pi^{128} \leftrightarrow dS_{34422} \, , \\
\pi^{129} \leftrightarrow dS_{13312} & , \cdots , & \pi^{144} \leftrightarrow dS_{34412} \, .
\end{array}$$

We assumed that all the dS_{abc11} can be solved for by the 16 exterior derivatives of the wave equation dW_{abc} and expressed the dS_{abc11} in this way.

An orderly ranking $\mathcal{R}_{\succ}^{wave,4}$ for the 160 second-order partial derivatives S_{abcde} for the Lanczos wave equation is based on the ordering

$$S_{abc44} \succ S_{abc34} \succ S_{abc33} \succ S_{abc24} \succ S_{abc23} \succ S_{abc22}$$

$$\succ S_{abc14} \succ S_{abc13} \succ S_{abc12} \succ S_{abc11}$$

and then amongst each set S_{abcij} for i, j fixed in the same way as for the P_{abci} in Appendix A, namely:

$$S_{344ij} \succ S_{343ij} \succ S_{243ij} \succ S_{234ij} \succ S_{244ij} \succ S_{242ij} \succ S_{233ij} \succ S_{232ij}$$

$$\succ S_{143ij} \succ S_{134ij} \succ S_{142ij} \succ S_{124ij} \succ S_{132ij} \succ S_{123ij} \succ S_{144ij} \succ S_{133ij} \, .$$

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[1] The Penrose wave equation for C_{abcd} can be found for all dimensions $n \leq 4$. It was not derived from the Lanczos wave equation, which was given for $n = 5$ by Edgar and Höglund [**?**], and we postpone the discussion of the 5-dimensional Weyl-Lanczos problem and the related tensor wave equation to another paper.